Numerical Investigation of the Burgers-Fisher and FitzHugh-Nagumo Equations by Temimi and Ansari Method (TAM)

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Abstract
In this research article, the novel Temimi and Ansari method (TAM) is presented to solve the nonlinear Burgers-Fisher and FitzHugh-Nagumo equations respectively. These nonlinear equations play predominant roles in the field of physical sciences, engineering, medicines and social sciences. Comparison of the results obtained by (TAM) is made with other semi-analytical methods to ascertain its efficiency and flexibility. The result revealed, the TAM is powerful, effective, less computationally time consuming and elegant.

Keyword: *Temimi and Ansari Method (TAM), Burgers-Fisher Equation, FitzHugh-Nagumo Equation, Semi-Analytical Iterative Technique.*

1. Introduction

In the field of physical Sciences and Engineering, there abound numerous nonlinear equations which usually contains a combination of partial derivatives and the dissipative term. These equations gained prominence because of their appearance in many scientific applications and physical phenomena Wazwaz (2009). Prominent among these equations includes the following: Burgers, Fishers, Korteweg-de Vries (KDV), Huxley, Schrödinger, Wave, Advection, Emden-Fowler, standard Lane-Emden and Pantograph. Others are the combination of two or more of the above methods, which includes the Burgers-Fisher, Burgers-Huxley, FitzHugh-Nagumo and Kuramoto-Sivashinsky respectively. Among the above, the Burger-Fisher and FitzHugh-Nagumo equations are important in that the former appears in the propagation of shock waves, fluid dynamics, heat conduction, number theory, elasticity, acoustic transmission and variation in vehicle density in high ways: Ismail et al (2004), Chandraker et al (2016), Garhane (2020), Behzadi and Araghi (2011), Javidi (2006), Kaya and El-Sayed (2003), Chandrasekaran and Ramasami (1996), Chen and Zhang (2004), Fahmy (2008), Lu et al (2007), Zhang and Yan (2010), whereas the latter finds useful application predominantly in mathematical finance, traffic flow, dynamics of gas, applied mathematics and applications in physics, Salmon et al (2016), FitzHugh-Nagumo (1961), Nagumo et al (1962), Rauch and Smoller (1978), McKean (1970). Principally, these equations describe the interaction between the reaction mechanism, diffusion, transport and convection coefficients respectively, Hodgkin and Huxley (1952)

Scientists and engineers have extensively solved these equations for closed form or approximate solutions using myriad of analytical and semi-analytical methods. Wazwaz (2005) used the tanh method to investigate the generalized forms of the nonlinear conduction and Burgers-Fisher equation. Using the spectral collocation and domain decomposition, Javidi (2006), Golbabai and Javidi (2009) examined the generalized Burger-Fisher equation. Equally, others explored the Burgers-Fisher equation using finite difference scheme Mikens and Gumel (2002), Chen (2007). Kocacoban et al (2011) gave elaborate, detailed and better approximation to the solutions of Burgers-Fisher equation using differential transform method. The result obtained was accurate, elegant and agree with results in literature. Similarly, the FitzHugh-Nagumo equation have also gained considerable attention from scholars the world over especially in the nonlinear sciences. The exact solution of the FitzHugh-Nagumo equation was studied using Homotopy perturbation method by Salmon et al (2016). Kawahara and Tanaka (1983) obtained the exact solution of the FitzHugh-Nagumo equation using Hirota method. Also, Nucci and Clarkson (1992) used the Jacobi elliptic function to solve and present new solutions of the FitzHugh-Nagumo equation. Soliman (2012) adopted the semi-analytical methods viz variational iteration and adomian decomposition methods to obtain the numerical solutions of the FitzHugh-Nagumo equations. Some recent methods for the numerical solution of time-dependent partial differential were studied by Gourlay (1971). Khalifa (1979) investigated the theory and applications of the collocation method with splines for ordinary and partial differential equations. Using Galerkin procedure, Cannon and Ewing (1977) studied parabolic systems of partial differential equations related to the transmission of nerve impulses.

In this research article, we apply the Temimi and Ansari method to solve both the Burgers-Fisher and FitzHugh-Nagumo equations. The result obtained agrees with those in literature and promising. The method handles the nonlinear equations, elegantly, comfortably and accurately with solutions which converges to the exact solution.

2. Temimi and Ansari Method (TAM)

Following Ebiwareme (2021), Al-Jawary and Mohammed (2015), we consider the general differential equation in operator form as follows

$$L\big(y(x)\big) + N\big(y(x)\big) + f(x) = 0, \quad x\epsilon D \tag{1}$$

$$B\left(y, \frac{dy}{dx}\right) = 0, \ x\epsilon\mu \tag{2}$$

Where x is the independent variable, $y(x)$ is an unknown function, $f(x)$ is a given known function, L is a linear operator, N is a nonlinear operator and B is a boundary operator.

To implement TAM, we first assume an initial guess of the form, $y_0(x)$ that satisfy the equation as follows

$$L\big(y_0(x)\big) + f(x) = 0, \ \ B\left(y_0, \frac{dy_0}{dx}\right) = 0 \tag{3}$$

The second iteration as follows

$$L\big(y_1(x)\big) + N\big(y_0(x)\big) + f(x) = 0,, B\left(y_1, \frac{dy_1}{dx}\right) = 0 \tag{4}$$

We consider the next iteration as follows

$$L\big(y_2(x)\big) + N\big(y_1(x)\big) + f(x) = 0,, B\left(y_2, \frac{dy_2}{dx}\right) = 0 \tag{5}$$

Continuing the same way, we obtain nth iterative procedure to give the subsequent iterates as

$$L\big(y_{n+1}(x)\big) + N\big(y_n(x)\big) + f(x) = 0, \ B\left(y_{n+1}, \frac{dy_{n+1}}{dx}\right) = 0 \tag{6}$$

From Eq. (6), each $y_i(x)$ is considered alone as a solution of Eq. (1). This method is easy to implement, straightforward and direct. The method gives better approximate solution which converges to the exact solution with only few members.

3. Numerical Evaluation

In this section, we apply the Temimi and Ansari method to solve various forms of the Burgers-Fisher and FitzHugh-Nagumo equations. Comparison is made between the efficiency of the method and show whether the obtained solution converges to the exact solution.

3.1 Burgers-Fisher Equation

Following Temimi and Ansari (2011a) and Temimi and Ansari (2011b), the Burgers-Fishers equation is given as follows

$$u_t - u_{xx} = uu_x + u(1-u) \tag{7}$$

Subject to the initial condition

$u(x,0) = 2x$ (8)

By applying TAM on both sides of Eq. (7), we obtain the linear and nonlinear operators as well as the known function as follows

$$L\big(u(x,t)\big) = u_t$$
$$N\big(u(x,t)\big) = -uu_x - u_{xx} - u - u^2$$
$$N\big(u(x,t)\big) = -(uu_x + u_{xx} + u - u^2)$$
$$f(x,t) = 0$$

The approximate solution of Eq. (7) takes the form

$L\big(u_0(x,t)\big) + f(x,t) = 0,\ \ u_0(x,t) = 2x$ (9)

Taking the inverse operator of both sides of Eq. (9) $w.r.t$, we obtain

$\int_0^t u_{0t}^1(x,t)dt = 0$ (10)

Solving Eq. (10) give the initial iterative solution as

$u_0(x,t) = 2x$ (11)

The second iterative solution is obtain using the relation

$L\big(u_1(x,t)\big) + N\big(u_0(x,t)\big) + f(x,t) = 0,\ u_1(x,t) = 2x$ (12)

Since $f(x,t) = 0$, Eq. (12) reduced to the form

$$L\big(u_1(x,t)\big) + N\big(u_0(x,t)\big) = 0$$

$L\big(u_1(x,t)\big) = -N\big(u_0(x,t)\big)$ (13)

Integrating both sides of Eq. (5) from 0 to t, we get

$\int_0^t u_{1t}^1(x,t)\,dt = \int_0^t [u_0(x,t)u_{0x}(x,t) + u_{0xx}(x,t) + u_0^2(x,t) - u_0(x,t)]dt$ (14)

Substituting Eq. (11) and taking the appropriate differentials into Eq. (14) gives

$$\int_0^t u_{1t}^1(x,t)dt = \int_0^t (6x - 4x^2)dt$$
$$u_1(x,t) = 2x + 6xt - 4x^2t$$

The third iterative solution of the problem is obtained using the form

$L\big(u_2(x,t)\big) + N\big(u_1(x,t)\big) + f(x,t) = 0,\ u_2(x,t) = 2x$ (15)

$$\Longrightarrow\ L\big(u_2(x,t)\big) = -N\big(u_1(x,t)\big)$$

Taking the inverse operator of both sides and integrating from the range 0 to t, we obtain

$\int_0^t u_{2t}^1(x,t)dt = \int_0^t [u_1(x,t)u_{1x}(x,t) + u_{1xx}(x,t) + u_1^2(x,t) - u_1(x,t)]dt$ (16)

Solving Eq. (16) gives the third iterate as

$$u_2(x,t) = 2x + 32x^3t^2 - 72x^2t^2 - 24x^2t + 36xt^2 + 24xt + 4x$$

Continuing in the same way, and using the identity

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)$$

Eq. (7) converges to the exact solution.

$$u(x,t) = 2x$$

3.2 FitzHugh-Nagumo Equation

In this subsection, we solve the Fitzhugh-Nagumo equation using the Temimi and Ansari method for different values of the control parameter, a

Recall the Fitzhugh-Nagumo equation as follows

$u_t = u_{xx} - u(1-u)(a-u),\ \ a > 0$ (17)

Rearranging Eq. (17) and write it in alternative form as

$u_t - u_{xx} = u(1-u)(u-a)$ (18)

where a is an arbitrary constant and $0 < a < 1$.

Clearly, when $a = -1$, Eq. (18) reduced to Newell-Whitehead equation

From Eq. (18), u_{xx} is the variation of the unknown with the spatial variable, x at a given time, whereas term on the right-hand side, $u(1-u)(u-a)$ denote the source term and $u(x,t)$ is a solution comprising the spatial variable, x and the temporary variable, t with $x\epsilon R$ and $t \geq 0$

To illustrate the workability, reliability and capability of the TAM on the FitzHugh-Nagumo equation, we consider two special cases when $a = 1, 2$

Case 1. In this case, we examine the FitzHugh-Nagumo equation by setting $a = 1$, the equation now become

$u_t - u_{xx} = u(1-u)(u-1)$ (19)

Subject to the initial condition, $u(x,0) = 2x$ (20)

Rearranging Eq. (19), it takes the form

$u_t - u_{xx} + u(1-u)^2 = 0$ (21)

Applying TAM to both sides of Eq. (21), we have the operators, L and N as follows

$L(u(x,t)) = u_t, \,, f(x,t) = 0,\ N(u(x,t)) = -[u_{xx} - u(1-u)]^2$

The first approximate solution of the problem become

$L(u_0(x,t)) + f(x,t) = 0$ (22)

$L(u_0(x,t)) = 0$ (23)

Integrating both sides of Eq. (23) from 0 to t gives

$\int_0^t u^1{}_{0t}(x,t)dt = 0$ (24)

$$u_0(x,t) - u(x,0) = 0$$

$u_0(x,t) = 2x$ (25)

The second iterative solution of the problem gives

$L(u_1(x,t)) + N(u_0(x,t)) + f(x,t) = 0, u_1(x,0) = 2x$ (26)

Since $f(x,t) = 0$, Eq. (26) reduce to

$$L(u_1(x,t)) = -N(u_0(x,t))$$

Integrating both sides of Eq. (26) form 0 to t, we obtain

$\int_0^t u_{1t}^1(x,t)dt = \int_0^t \left[u_{0xx}(x,t) - u_0(x,t)\left(1-u_0(x,t)\right)^2\right] dt$ (27)

$$\int_0^t u_{1t}^1(x,t)dt = \int_0^t [-2x(1-4x+4x^2)]dt$$

Simplification of the above gives

$u_1(x,t) = 2x - 2xt + 8x^2t - 8x^3t$ (28)

The third iterative solution of the problem gives

$L(u_2(x,t)) + N(u_1(x,t)) + f(x,t) = 0, u_2(x,0) = 2x$ (29)

$$L(u_2(x,t)) + N(u_1(x,t)) = 0$$

$L(u_2(x,t)) = -N(u_1(x,t)),\ u_2(x,0) = 2x$ (30)

Integrating both sides of Eq. (30) from 0 to t gives

$\int_0^t u_{2t}^1(x,t)dt = \int_0^t \left[u_{1xx}(x,t) - u_1(x,t)\left(1-u_1(x,t)\right)^2\right] dt$ (31)

Simplification of Eq. (31) gives the third iterative solution as

$$u_2(x,t) = 2x + 20t - 4t^2 - 80tx + 64t^2x + 48tx^2 - 352t^2x^2 + 768t^2x^3 - 576t^2x^4$$

Continuing in the same way, the solution converges to the exact solution using the identity,

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)$$

$$u(x,t) = 2x(1 - t + 4xt - 4x^2t + \cdots)$$

$u(x,t) = \frac{2x}{1+2xt}$ (32)

Case II**:** Putting $a = 2$ into Eq. (18), the FitzHugh-Nagumo equation become

$u_t - u_{xx} = u(1-u)(u-2)$ (33)

Subject to the initial condition, $u(x,0) = 2x$ (34)

Applying TAM to both sides of Eq. (33), we get

$$L(u(x,t)) = u_t = \frac{\partial u}{\partial t}$$

$$N(u(x,t)) = -u_{xx} + u(u-1)(u-2)$$

$$f(x,t) = 0$$

The first iterative solution of the problem becomes

$L(u_0(x,t)) + f(x,t) = 0,\ u_0(x,0) = 2x$ (35)

Integrating both sides of the equation from 0 to t, we obtain

$$\int_0^t u_{0t}^1(x,t)dt = 0$$

$u_0(x,t) = 2x$ (36)

The second iterative solution of Eq. (33) becomes

$L(u_1(x,t)) + N(u_0(x,t)) + f(x,t) = 0, u_1(x,0) = 2x$ (37)

Since $f(x,t) = 0$

$L(u_1(x,t)) = -N(u_0(x,t))$ (38)

Integrating both sides of Eq. (38) and taking the limits from 0 to t gives

$\int_0^t u_{1t}^1(x,t)\,dt = \int_0^t [u_{0xx}(x,t) + u_0(x,t)(u_0(x,t)-1)(u_0(x,t)-2)]dt$ (39)

$$\int_0^t u_{1t}^1(x,t)dt = \int_0^t [2x(2x-1)(2x-2)]dt$$

$$u_1(x,t) - 2x = \int_0^t (8x^3 - 12x^2 + 2x)dt$$

$\Rightarrow \quad u_1(x,t) = 2x + 8x^3t - 12x^2t + 2xt$

$\Rightarrow u_1(x,t) = 2x(1 + t + 4tx^2 - 6xt)$ (40)

The third iterative solution of the problem similarly gives

$L(u_2(x,t)) + N(u_1(x,t)) + f(x,t) = 0, u_2(x,0) = 2x$ (41)

On rearranging Eq. (41), we get

$L(u_2(x,t)) = -N(u_1(x,t))$ (42)

Integrating both sides of Eq. (42), and taking the limits from 0 to t, we obtain

$\int_0^t u_{2t}^1(x,t)dt = \int_0^t [u_{1xx}(x,t) + u_1(x,t)(u_1(x,t)-1)(u_1(x,t)-2)]dt$ (43)

Simplification of Eq. (43), we obtain the iterative solution as

$u_2(x,t) = 2x$

4. Concluding Remarks

In this paper, the application of the novel Temimi and Ansari method to the numerical solution of the nonlinear Burgers-Fisher and FitzHugh-Nagumo equations is theoretically investigated. The solutions of these equations are obtained after a finite number of iterations and converges rapidly to the exact solution. The result obtained shows this method is satisfactory, powerful, efficient and provides validity with potential to solve other nonlinear differential and partial differential equations.

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